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JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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Technical Memorandum No. 33-15

The Statistical Analysis of Space Guidance Systems

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CONTENTS

I. INTRODUCTION

In analyzing the errors of guidance systems, JPL experience has indicated that the treatment of injection-errors as independent random variables may sometimes be a poor approximation. Statistical calculations of dispersions at the destination, or of the magnitude of a midcourse maneuver, must take into account cross-correlations between injection errors; the appropriate theory is presented in this Memorandum.

The analysis hinges on the properties of multivariate Gaussian distributions and linear perturbation theory. Referring to the latter, the validity of the approximations of first-order error analyses can only be confirmed by testing the linearity of relevant perturbations for each particular case. No attempt has therefore been made to formulate guide lines as to when first-order approximations are acceptable and when they are not. It is sufficient to say that, in the cases analyzed to date at JPL, the application of perturbation theory was amply justified.

The theory is presented with reference to the problem of a single-stage guided rocket, after which consideration is given to multi-stage ascents, satellite orbits, calculation of the average midcourse maneuver and of the dispersion at a destination in the absence of post-injection guidance.

Section II provides the introduction of the appropriate theory, with reference to a simple example, viz. the ascent of a guided single-stage rocket. The extension to multistage rockets and the calculation of the dispersion in orbital parameters is then presented in Section III.

II. THE STATISTICAL DESCRIPTION OF INJECTION ERRORS

Consider the case of a single-stage rocket the ascent of which is controlled by some guidance system. It is not necessary to specify what form of guidance but it may be helpful to think of the guidance being as follows: the attitude (and hence the thrust vector) of the vehicle is slaved to a (gyro) reference direction which is pitched over at some predetermined rate. An accelerometer is mounted with its sensitive axis along the axis of the vehicle, and when the integrated output of the accelerometer attains a certain value the rocket motor is shut off. Without becoming involved in a detailed analysis of such a guidance system, some sources of error may be mentioned.

a. The thrust axis may not be along the vehicle axis so that, unknown to the autopilot, the vehicle is forced to fly on a trajectory which differs slightly from the standard.

b. Gyroscope drift (apart from the programmed precession) may also bring about nonstandard trajectories.

c. A null-shift in the accelerometer may lead to an error in the computed axial velocity.

d. The rocket motor may develop more or less thrust than the nominal value, with the result that the motor is shut off with the right axial velocity but in the wrong position, and so forth.

In nearly all cases which arise in practice, nonstandard trajectories have two important characteristics: (1) the source of error is a random quantity distributed approximately according to the Gaussian Law (e.g. accelerometer null-shift)[1] and (2) the trajectories which are in error differ only slightly from the standard, so that linear perturbation theory may be invoked to calculate such deviations.

The effect of one error source is first examined and it is assumed that, for each kind of error, the trajectory of the vehicle can be computed either on a digital machine or analytically. Let the error be δe, which may be thought of as any one of the abovementioned errors, and let δx, δy, δz, δu, δv, δw be the resulting perturbations in position and velocity at a fixed time after launch. The time is chosen to be shortly after the standard burnout point but sufficiently late so that, on perturbed trajectories, burnout always occurs before the fixed time. To a first order of approximation (perturbation theory):

$$\left.\begin{aligned} \delta x &= a_1\,\delta e \\ \delta y &= a_2\,\delta e \\ &\cdots\cdots\cdots \\ \delta w &= a_6\,\delta e \end{aligned}\right\} \quad (1)$$

The a_i coefficients are calculated analytically or by introducing the error δe in a computing program which simulates (analog or digital) the vehicle, propulsion and guidance systems.

The error δe is a random variable in the ensemble generated by an infinite number of hypothetical ascents. Furthermore, since δe is normally distributed, so are the six injection coordinates of Eq. (1)[2]. However, these coordinates are not characterized by six independent normal distributions; instead they are described by a six-dimensional *joint* distribution. Thus it is shown in statistical text books (e.g., Cramér) that the probability of the injection coordinates being between x and $x + dx$, y and $y + dy$, etc., is

$$\frac{1}{(2\pi)^{3}\,(|M|)^{1/2}} \exp\left\{-\tfrac{1}{2}\,\delta X^{T} M^{-1} \delta X\right\} dx\,dy \ldots dw \quad (2)$$

where

$$\delta X = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta u \\ \delta v \\ \delta w \end{bmatrix} \quad (3)$$

and M is the moment matrix (covariance matrix) of the six-dimensional distribution. Knowledge of M gives a complete statistical description of the injection coordinates. It is calculated as follows:

$$\text{Put } \delta X = A\,\delta e \quad (4)$$

where

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \quad (5)$$

[1]This theory is restricted to the important class of unpredictable error-sources which are constant for one flight but which are random variables when comparing one instrument or rocket motor to another.

[2]Mathematical Methods of Statistics, by H. Cramér, Princeton University Press, 1957, page 313.

then

$$\delta X \delta X^T = A(\delta e)^2 A^T \tag{6}$$

The ensemble average is now taken of all terms in the matrices on both sides of Eq. (6). This gives the moment matrix of injection errors

$$M = \overline{\delta X \delta X^T} = A\overline{(\delta e)^2} A^T \tag{7}$$

Written out in full, the result is

$$M = \begin{bmatrix} \overline{\delta x^2} & \overline{\delta x \delta y} & \ldots & \overline{\delta x \delta w} \\ \overline{\delta y \delta x} & \overline{\delta y^2} & \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \overline{\delta w \delta x} & \ldots & \ldots & \overline{\delta w^2} \end{bmatrix} = \sigma^2 AA^T \tag{8}$$

where σ is the standard deviation of the error δe. It will be seen that M is a symmetrical matrix; the diagonal terms are the variances on the six injection errors and the other terms are covariances. In order to introduce cross-correlation coefficients it is sometimes convenient to write equation (8) in the form

$$M = \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y & \rho_{xz}\sigma_x\sigma_z & \ldots & \rho_{xw}\sigma_x\sigma_w \\ & \sigma_y^2 & \rho_{yz}\sigma_y\sigma_z & \ldots & \ldots \\ \text{(symmetric)} & & & & \vdots \\ & & & & \\ & & & & \sigma_w^2 \end{bmatrix} \tag{8}$$

where σ_x is the rms injection error in x and ρ_{xy} is the coefficient of cross-correlation between x and y, etc. The coefficient $\rho = \pm 1$ for perfectly correlated variables and $\rho = 0$ for uncorrelated variables.

Crude analyses of guidance systems have often ignored correlation effects, a poor approximation since some injection variables are invariably heavily correlated, e.g., speed and height.

The calculation of the moment matrix is now generalized to include more than one error source, e.g., accelerometer null-shift, gyro drift, under or over-performing rocket motor. Let the errors due to n sources be δe_i ($i = 1, 2 \ldots n$). Then, assuming that such errors perturb the ascent trajectory only slightly, the injection error may be written as linear combinations of the errors due to each source.

$$\left.\begin{aligned} \delta x &= a_{11}\,\delta e_1 + a_{12}\,\delta e_2 + \ldots + a_{1n}\,\delta e_n \\ \delta y &= a_{21}\,\delta e_1 + a_{22}\,\delta e_2 + \ldots + a_{2n}\,\delta e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta w &= a_{61}\,\delta e_1 + a_{62}\,\delta e_2 + \ldots + a_{6n}\,\delta e_n \end{aligned}\right\} \tag{9}$$

In matrix notation, Eq. (9) is

$$\delta X = A\delta E \tag{10}$$

where

$$\delta E = \begin{bmatrix} \delta e_1 \\ \delta e_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta e_n \end{bmatrix} \tag{11}$$

and A is the $(6 \times n)$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{61} & a_{62} & \ldots & a_{6n} \end{bmatrix} \tag{12}$$

The $6n$ terms of the A matrix would again be computed either analytically or by perturbing trajectories computed on a machine. For n error-sources the A matrix could be calculated by computing n trajectories.

Provided that the error-sources are random quantities with Gaussian distributions, the six injection-errors will once more have a six-dimensional joint Gaussian distribution defined by Eq. (2). The calculation of the moment matrix is now as follows:

From Eq. (10)

$$\delta X \delta X^T = A\delta E \delta E^T A^T \tag{13}$$

and by taking the ensemble average

$$M = \overline{\delta X \delta X^T} = A\overline{\delta E \delta E^T} A^T \tag{14}$$

The $(n \times n)$ symmetric matrix $\overline{\delta E \delta E^T}$ $(= \Lambda)$ is the moment matrix of the error-sources, but in nearly all cases these can be considered as independent random variables, e.g., gyro drift is independent of rocket motor performances. In this case the matrix is diagonal:

$$\overline{\delta E \delta E^T} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & \ldots & & \\ 0 & \sigma_2^2 & 0 & 0 & \ldots & & \\ 0 & 0 & \sigma_3^2 & 0 & \ldots & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \\ \ldots & & & & \sigma_{n-1}^2 & 0 \\ & & & & 0 & \sigma_n^2 \end{bmatrix} \tag{15}$$

the n diagonal terms being the variances of the n error-sources.

The moment matrix of the injection errors can therefore be written

$$M = A\Lambda A^T \tag{16}$$

where in most cases Λ is the diagonal matrix of Eq. (15).

III. APPLICATION TO SATELLITE AND SPACE MISSIONS

The procedure derived in Section II was developed with a single-stage rocket in mind. The same method can, however, be applied to multistage ascents and for calculating post-injection dispersions.

A. Injection Errors of a Multistage Ascent

In a multistage ascent, errors at first burnout are propagated through subsequent stages and mixed with errors in those stages. A possible difficulty is that some, but not all, of the errors developed in separate stages are correlated. For example, although with separate rocket motors for each stage the performance of a motor is independent of its predecessor, the drift in one certain gyroscope may affect all stages in a similar manner.

Nevertheless, even with such complications, the computation of the moment matrix of injection errors (burnout of the last stage) can be carried out in the fashion explained in the preceding section. Referring to Eq. (9) and (10), the n error-sources include those in all stages, although if one error-source creates errors in more than one stage it is counted only once (e.g., gyro drift in a platform used for guiding all stages). The effect on injection of a motor-performance variation in the first stage would be computed by assuming ideal performance from the motors and guidance systems of subsequent stages. The effect on injection of a final-stage motor variation would be computed assuming the standard ascent trajectory up to the last stage. The effect of drift in a gyroscope common to all stages would be computed by simulating that drift in all stages but assuming no other error sources, and so on.

The formal calculation of the injection moment matrix is therefore just the same as for the single-stage rocket and Eq. (9) through (19) apply. Some numerical examples of JPL analyses are included in Section III-C below; such calculations included the effect a long coast (20 minutes) between burnout of the second stage and ignition of the third stage.

B. Dispersion of Earth-Satellite Orbital Parameters

Suppose that a satellite has been placed into orbit and, by the methods discussed above, the moment matrix of injection errors has been evaluated. The calculation of the dispersion in the orbital parameters is now considered. The six parameters commonly employed are as follows:

a = semi-major axis (or the period)

e = eccentricity

i = inclination of the plane of the orbit to the equatorial plane

Ω = longitude of the node

ω = argument of the perigee

α = mean anomaly at epoch

For highly eccentric orbits these parameters would be acceptable, but for near-circular orbits, perturbations in the eccentricity e are not linearly related to perturbations in the injection coordinates ($e > 0$, except for a circular orbit where $e = 0$). For this reason an alternative parameter is recommended, viz. the difference between the apogee R_a and perigee R_p. By definition of these quantities $\rho = (R_a - R_p)$ is always positive but by taking

$$\left| \frac{\partial}{\partial X} (R_a - R_p) \right|$$

(where X is an injection coordinate) the difficulty is avoided[1]. Instead of eccentricity e, the parameter ρ ($= R_a - R_p$) will therefore be used.

Assuming linear perturbation theory to be valid, the six injection errors of Eq. (3) are linearly related to errors in the orbital parameters. Thus

$$\begin{bmatrix} \delta a \\ \delta \rho \\ \delta i \\ \delta \Omega \\ \delta \omega \\ \delta \alpha \end{bmatrix} = C \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta u \\ \delta v \\ \delta w \end{bmatrix} \tag{17}$$

where the elements of the (6×6) matrix C may be calculated analytically or by machine. Presumably the (x, y, z) axes would be chosen with x as the local vertical and y in the horizontal and orbital planes.

The injection errors satisfy a six-dimensional normal distribution and, since the errors in the orbital parameters are linearly related to the injection errors, they will also satisfy a six-dimensional normal distribution with the following moment matrix:

[1] JPL Internal Technical Memo No. 528, by W. Kizner.

$$\begin{bmatrix} \overline{\delta a^2} & \overline{\delta a \delta p} & \overline{\delta a \delta i} & \overline{\delta a \delta \Omega} & \overline{\delta a \delta \omega} & \overline{\delta a \delta a} \\ & \overline{\delta p^2} & \overline{\delta p \delta i} & \overline{\delta p \delta \Omega} & \overline{\delta p \delta \omega} & \overline{\delta p \delta a} \\ & & \overline{\delta i^2} & \overline{\delta i \delta \Omega} & \overline{\delta i \delta \omega} & \overline{\delta i \delta a} \\ & & & \overline{\delta \Omega^2} & \overline{\delta \Omega \delta \omega} & \overline{\delta \Omega \delta a} \\ & \text{(symmetric)} & & & \overline{\delta \omega^2} & \overline{\delta \omega \delta a} \\ & & & & & \overline{\delta a^2} \end{bmatrix} = CMC^T \tag{18}$$

M being the moment matrix of injection errors. Usually, only the rms errors in the orbital parameters are of interest, i.e., the diagonal terms of CMC^T.

C. Calculation of the Average Midcourse Maneuver

If a midcourse maneuver is to be applied, the magnitude of the maneuver depends on the size and cross-correlations of the injection errors. Given the guidance system and the moment matrix of injection errors, it is important to be able to estimate the required midcourse maneuver. For a given midcourse point on a given trajectory (lunar or interplanetary) it can be shown[1] that, to a first order of approximation, the three components of the maneuver are given by

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = K \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta u \\ \delta v \\ \delta w \end{bmatrix} = K\delta X \tag{19}$$

where K is a (3×6) matrix consisting of elements computed on the standard trajectory.

It follows that

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \cdot [V_x V_y V_z] = K\delta X \delta X^T K^T \tag{20}$$

and by taking the ensemble average

$$\begin{bmatrix} \overline{V_x^2} & \overline{V_x V_y} & \overline{V_x V_z} \\ \overline{V_y V_x} & \overline{V_y^2} & \overline{V_y V_z} \\ \overline{V_z V_x} & \overline{V_z V_y} & \overline{V_z^2} \end{bmatrix} = KMK^T \tag{21}$$

where M is the moment matrix of injection errors. KMK^T is the moment matrix of the three-dimensional normal distribution of the components V_x, V_y, V_z. When KMK^T has been computed numerically the sum of the three diagonal terms gives

$$\overline{V_x^2} + \overline{V_y^2} + \overline{V_z^2} = \overline{(V_x^2 + V_y^2 + V_z^2)} \tag{22}$$

i.e., the mean-square value of the midcourse maneuver.

For approximate analyses, it may be sufficient to know the rms value of the maneuver; it may be reasoned that, if a midcourse rocket can deliver three times the rms value, then it can cope with a very high proportion of all possible cases. However, although V_x, V_y and V_z satisfy a joint Gaussian distribution, the distribution of $(V_x^2 + V_y^2 + V_z^2)^{1/2}$ is *not* Gaussian. The calculation of the distribution function of the magnitude of the maneuver is beyond the scope of this paper.

D. The Dispersion Ellipses at the Moon or the Planets

In the absence of post-injection guidance the dispersion at the destination is a function of the choice of the trajectory and the errors at injection. The dispersion may be measured in terms of miss components on the surface of the moon or miss components at the planets measured in a plane perpendicular to the approach velocity vector. Whatever the coordinates employed, the two components of miss will be referred to as M_1 and M_2.

From perturbation calculations on the standard trajectory a U matrix can be formed such that

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = U\delta X \tag{23}$$

U being a (2×6) matrix. Now

$$\begin{bmatrix} M_1^2 & M_1 M_2 \\ M_2 M_1 & M_2^2 \end{bmatrix} = U\delta X \delta X^T U^T \tag{24}$$

and, by taking the ensemble average,

$$\begin{bmatrix} \overline{M_1^2} & \overline{M_1 M_2} \\ \overline{M_2 M_1} & \overline{M_2^2} \end{bmatrix} = UMU^T \tag{25}$$

i.e., the moment matrix of the two dimensional Gaussian distribution for the two miss components. Put

$$UMU^T = \begin{bmatrix} a & b \\ b & b \end{bmatrix} \tag{26}$$

then

$$(UMU^T)^{-1} = \frac{1}{(ab - b^2)} \begin{bmatrix} b & -b \\ -b & a \end{bmatrix} \tag{27}$$

and the two-dimensional probability density function is (cf. Eq. 2):

$$\frac{1}{2\pi (ab - b^2)^{1/2}} \exp\left\{ -\frac{1}{2(ab - b^2)} (bM_1^2 - 2bM_1 M_2 + aM_2^2) \right\} \tag{28}$$

[1] Interplanetary Post-Injection Guidance, by A. R. M. Noton. JPL External Publication No. 653, 1959, page 34.

It will be observed that contours of constant probability are ellipses in the (M_1, M_2) plane,

$$bM_1^2 - 2b\, M_1M_2 + a\, M_2^2 = \text{constant} \qquad (29)$$

It can be shown that, for such an ellipse having semi-minor and semi-major axes $k\lambda_1$ and $k\lambda_2$ respectively, where

$$\left.\begin{aligned} \lambda_1^2 &= \tfrac{1}{2}(a+b) + \left[\left(\frac{a-b}{2}\right)^2 + h^2\right]^{\frac{1}{2}} \\ \lambda_2^2 &= \tfrac{1}{2}(a+b) - \left[\left(\frac{a-b}{2}\right)^2 + h^2\right]^{\frac{1}{2}} \end{aligned}\right\} \qquad (30)$$

the probability of the miss being within such an ellipse is[5]

$$P = 1 - e^{-\frac{1}{2}k^2} \qquad (31)$$

(when $k = 1$, 2 and 3, $P = 0.40$, 0.86 and 0.99 respectively.) These ellipses have the major axis inclined at an angle θ to the M_1 axis, where

$$\theta = \tfrac{1}{2}\tan^{-1}\frac{2b}{a-b} \qquad (32)$$

As a numerical example, some results of JPL analyses are quoted. It should be noted that the numbers do not refer to an existing or anticipated guidance system. The coordinates are defined in the sketch, the y-axis being the local vertical at the standard injection point. Table 1 presents the results for three slightly different guidance systems[6]; the table includes rms errors, cross-correlation coefficients (Eq. 8) as deduced from the moment matrix of injection errors, and the parameters of the dispersion ellipse.

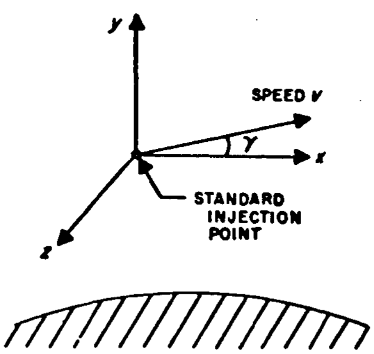

Table 1. 66-hour lunar impact trajectory.

Group	Parameter	System 1	System 2	System 3
rms injection errors	σ_x, ft	15,900	13,700	13,500
	σ_y, ft	14,800	10,200	10,000
	σ_v, ft/sec	15.6	11.7	11.5
	σ_γ, millirad	0.733	0.515	0.508
	σ_z, ft	41,300	41,300	40,200
	$\sigma_{\dot z}$, ft/sec	152	152	96
cross-correlation coefficients of injection errors	ρ_{xy}	−0.715	−0.902	−0.910
	ρ_{xv}	0.794	0.917	0.916
	$\rho_{x\gamma}$	−0.700	−0.823	−0.829
	ρ_{yv}	−0.490	−0.969	−0.984
	$\rho_{y\gamma}$	0.952	0.904	0.905
	$\rho_{v\gamma}$	−0.481	−0.872	−0.887
	$\rho_{z\dot z}$	0.753	0.753	0.934
99% probability ellipse	semi-major axis, mi.	7,770	2,300	2,250
	semi-minor axis, mi.	1,710	910	706

Inspection of the table reveals that, if the guidance system were judged merely on the basis of rms injection-errors, then System 3 would appear to be only a little better than System 1. However, it will be observed that the dispersion ellipse of System 3 has only 0.29 the length of that of System 1. The relative accuracy of System 3 is mainly attributable to the high correlation coefficients; errors are tending to compensate for each other, e.g., height and speed are almost perfectly correlated in System 3. The effects of such correlations can be taken into account only by the procedures outlined in this Memorandum.

[5]Prediction of Ballistic Missile Trajectories from Radar Observations, by I. Shapiro, McGraw-Hill Book Co., 1958, page 87.

[6]From unpublished studies by C. G. Pfeiffer, A. Dickinson, and C. E. Kohlhase.